

Problem 2.14

[Difficulty: 2]

2.14 Beginning with the velocity field of Problem 2.5, show that the parametric equations for particle motion are given by $x_p = c_1 e^{At}$ and $y_p = c_2 e^{-At}$. Obtain the equation for the pathline of the particle located at the point $(x, y) = (2, 2)$ at the instant $t = 0$. Compare this pathline with the streamline through the same point.

Given: Velocity field

Find: Proof that the parametric equations for particle motion are $x_p = c_1 e^{At}$ and $y_p = c_2 e^{-At}$; pathline that was at $(2, 2)$ at $t = 0$; compare to streamline through same point, and explain why they are similar or not.

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = A \cdot x$ $v_p = \frac{dy}{dt} = -A \cdot y$

So, separating variables $\frac{dx}{x} = A \cdot dt$ $\frac{dy}{y} = -A \cdot dt$

Integrating $\ln(x) = A \cdot t + C_1$ $\ln(y) = -A \cdot t + C_2$
 $x = e^{A \cdot t + C_1} = e^{C_1} \cdot e^{A \cdot t} = c_1 \cdot e^{A \cdot t}$ $y = e^{-A \cdot t + C_2} = e^{C_2} \cdot e^{-A \cdot t} = c_2 \cdot e^{-A \cdot t}$

The pathlines are $x = c_1 \cdot e^{A \cdot t}$ $y = c_2 \cdot e^{-A \cdot t}$

Eliminating t $t = \frac{1}{A} \cdot \ln\left(\frac{x}{c_1}\right) = -\frac{1}{A} \cdot \ln\left(\frac{y}{c_2}\right)$ $\ln\left(x^{\frac{1}{A}} \cdot y^{\frac{1}{A}}\right) = \text{const}$ or $\ln(x^A \cdot y^A) = \text{const}$

so $x^A \cdot y^A = \text{const}$ or $x \cdot y = 4$ for given data

For streamlines $\frac{v}{u} = \frac{dy}{dx} = -\frac{A \cdot y}{A \cdot x} = -\frac{y}{x}$

So, separating variables $\frac{dy}{y} = -\frac{dx}{x}$

Integrating $\ln(y) = -\ln(x) + c$

The solution is $\ln(x \cdot y) = c$ or $x \cdot y = \text{const}$ or $x \cdot y = 4$ for given data

The streamline passing through $(2, 2)$ and the pathline that started at $(2, 2)$ coincide because the flow is steady!